Geophysical modelling with Colombeau functions: Microlocal properties and Zygmund regularity

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Abstract

In global seismology Earth's properties of fractal nature occur. Zygmund classes appear as the most appropriate and systematic way to measure this local fractality. For the purpose of seismic wave propagation, we model the Earth's properties as Colombeau generalized functions. In one spatial dimension, we have a precise characterization of Zygmund regularity in Colombeau algebras. This is made possible via a relation between mollifiers and wavelets.

1 Introduction

Wave propagation in highly irregular media. In global seismology, (hyperbolic) partial differential equations the coefficients of which have to be considered generalized functions; in addition, the source mechanisms in such application are highly singular in nature. The coefficients model the (elastic) properties of the Earth, and their singularity structure arises from geological and physical processes. These processes are believed to reflect themselves in a multi-fractal behavior of the Earth's properties. Zygmund classes appear as the most appropriate and systematic way to measure this local fractality (cf. [2, Chap.4]).

The modelling process and Colombeau algebras. In the seismic transmission problem, the diagonalization of the first order system of partial differential equations and the transformation to the second order wave equation requires differentiation of the coefficients. Therefore, highly discontinuous coefficients will appear naturally although the original model medium varies continuously. However, embedding the fractal coefficient first into the Colombeau algebra ensures the equivalence after transformation and yields unique solvability if the regularization scaling γ is chosen appropriately (cf. [7, 10, 4]). We use the frame-

work and notation (in particular, $\mathcal{G}$ for the algebra and $\mathcal{A}_N$ for the mollifier sets) of Colombeau algebras as presented in [11].

An interesting aspect of the use of Colombeau theory in wave propagation is that it leads to a natural control over and understanding of 'scale'. In this paper, we focus on this modelling process.

2 Basic definitions and constructions

2.1 Review of Zygmund spaces

We briefly review homogeneous and inhomogeneous Zygmund spaces, $\dot{C}^s_*(\mathbb{R}^m)$ and $C^s_*(\mathbb{R}^m)$, via a characterization in pseudodifferential operator style which follows essentially the presentation in [3], Sect. 8.6. Alternatively, for practical and implementation issues one may prefer the characterization via growth properties of the discrete wavelet transform using orthonormal wavelets (cf. [8]).

Classically, the Zygmund spaces were defined as extension of Hölder spaces by boundedness properties of difference quotients. Within the systematic and unified approach of Triebel (cf. [13, 15]) we can simply identify the Zygmund spaces in a scale of inhomogeneous and homogeneous (quasi) Banach spaces, B^s_{pq} and $\dot{B}^s_{pq}$ ($s \in \mathbb{R}$, $0 < p, q \leq \infty$), by $C^s_*(\mathbb{R}^m) = B^s_{\infty\infty}(\mathbb{R}^m)$ and $\dot{C}^s_*(\mathbb{R}^m) = \dot{B}^s_{\infty\infty}$. Both $C^s_*(\mathbb{R}^m)$ and $\dot{C}^s_*(\mathbb{R}^m)$ are Banach spaces.

To emphasize the close relation with mollifiers we describe a characterization of Zygmund spaces in pseudodifferential operator style in more detail.

Let $0 < a < b$ and choose $\varphi_0 \in \mathcal{D}(\mathbb{R})$, φ_0 symmetric and positive, $\varphi_0(t) = 1$ if $|t| < a$, $\varphi_0(t) = 0$ if $|t| > b$, and φ_0 strictly decreasing in the interval (a, b). Putting $\varphi(\xi) = \varphi_0(|\xi|)$ for $\xi \in \mathbb{R}^m$ then defines a function $\varphi \in \mathcal{D}(\mathbb{R}^m)$. Finally we set

$$\psi(\xi) = -\langle \xi | \operatorname{grad} \varphi(\xi) \rangle$$

and note that if $a < |\xi| < b$ then $\psi(\xi) = -\varphi_0'(|\xi|)|\xi| > 0$. We denote by $\mathcal{M}(\mathbb{R}^m)$ the set of all pairs $(\varphi, \psi) \in \mathcal{D}(\mathbb{R}^m)^2$ that are constructed as above (we usually suppress the dependence of $\mathcal{M}$ on a and b in the notation).

We are now in aposition to state the characterization theorem for the inhomogeneous Zygmund spaces as subspaces of $\mathcal{S}'(\mathbb{R}^m)$. It follows from [14], Sec. 2.3, Thm. 3 or, alternatively, from [3], Sec. 8.6. Note that all appearing pseudodifferential operators in the following have x-independent symbols and are thus given simply by convolutions.

Theorem 1. Assume that $a \leq 1/4$ and $b \geq 4$ and choose $(\varphi, \psi) \in \mathcal{M}(\mathbb{R}^m)$ arbitrary. Let $s \in \mathbb{R}$ then $u \in \mathcal{S}'(\mathbb{R}^m)$ belongs to the inhomogeneous Zygmund

space of order s $C^s_*(\mathbb{R}^m)$ if and only if

$$(1)\qquad |u|_{C^s_*} := \|\varphi(D)u\|_{\mathrm{L}^\infty} + \sup_{0<t<1}\Big(t^{-s}\|\psi(tD)u\|_{\mathrm{L}^\infty}\Big) < \infty.$$

(Note that we made use of the modification for $q=\infty$ in [14], equ. (82).)

Remark 2. (i) $|u|_{C^s_*}$ defines an equivalent norm on C^s_*. In fact that all norms defined as above by some $(\varphi,\psi)\in\mathcal{M}(\mathbb{R}^m)$ are equivalent can be seen as in [3], Lemma 8.6.5.

(ii) If $s\in\mathbb{R}_+\setminus\mathbb{N}$ then $C^s_*(\mathbb{R}^m)$ is the classical Hölder space of regularity s. Denoting by $\lfloor s\rfloor$ the greatest integer less than s it consists of all $\lfloor s\rfloor$ times continuously differentiable functions f such that $\partial^\alpha f$ is bounded when $|\alpha|\le\lfloor s\rfloor$ and globally Hölder continuous with exponent $s-\lfloor s\rfloor$ if $|\alpha|=\lfloor s\rfloor$.

(iii) Due to the term $\|\varphi(D)u\|_{\mathrm{L}^\infty}$ the norm $|u|_{C^s_*}$ is not homogeneous with respect to a scale change in the argument of u.

(iv) If $u\in\mathrm{L}^\infty(\mathbb{R}^m)$ then (cf. [3], Sect. 8.6)

$$(2)\qquad u(x)=\varphi(D)u(x)+\int_1^\infty \psi(D/t)u(x)\frac{dt}{t}\qquad \text{for almost all } x.$$

Using $\varphi(\xi)=\int_0^1\psi(\xi/t)/t\,dt$ this can be rewritten in the form $u(x)=\int_0^\infty\psi(D/t)u(x)/t\,dt$ and resembles Calderon's classical identity in terms of a continuous wavelet transform (cf. [8], Ch. 1, (5.9) and (5.10)).

(v) In a similar way one can characterize the homogeneous Zygmund spaces as subspaces of $\mathcal{S}'(\mathbb{R}^m)$ modulo the polynomials $\mathcal{P}$. A proof can be found in [12], Sec. 3.1, Thm. 1. We may identify $\mathcal{S}'/\mathcal{P}$ with the dual space $\mathcal{S}_0'(\mathbb{R}^m)$ of $\mathcal{S}_0(\mathbb{R}^m)=\{f\in\mathcal{S}(\mathbb{R}^m)\mid \partial^\alpha\widehat{f}(0)=0\,\forall\alpha\in\mathbb{N}_0^m\}$, the Schwartz functions with vanishing moments, by mapping the class $u+\mathcal{P}$ with representative $u\in\mathcal{S}'$ to $u\,|_{\mathcal{S}_0}$. Assume that $a\le 1/4$ and $b\ge 4$ and choose $\psi\in\mathcal{D}(\mathbb{R}^m)$ as constructed above and let $s\in\mathbb{R}$ and $u\in\mathcal{S}'(\mathbb{R}^m)$. Then $u|_{\mathcal{S}_0}$ belongs to the homogeneous Zygmund space $\dot{C}^s_*(\mathbb{R}^m)$ of order s if and only if

$$(3)\qquad |u|_{\dot{C}^s_*} := \sup_{0<t<\infty}\Big(t^{-s}\|\psi(tD)u\|_{\mathrm{L}^\infty}\Big)<\infty.$$

(Note that we use the modification for $q=\infty$ in [12], equ. (16).)

2.2 The continuous wavelet transform

Following [2] we call a function $g\in\mathrm{L}^1(\mathbb{R}^m)\cap\mathrm{L}^\infty(\mathbb{R}^m)$ with $\int g=0$ a *wavelet*. We shall say that it is a *wavelet of order k* ($k\in\mathbb{N}_0$) if the moments up to order k vanish, i.e., $\int x^\alpha g(x)dx=0$ for $|\alpha|\le k$.

The (continuous) wavelet transform is defined for $f \in \mathrm{L}^p(\mathbb{R}^m)$ $(1 \le p \le \infty)$ by $(\varepsilon > 0)$

$$W_g f(x,\varepsilon) = \int\limits_{\mathbb{R}^m} f(y) \frac{1}{\varepsilon^m} \bar{g}(\frac{y-x}{\varepsilon})\, dy = f * (\check{\bar{g}})_\varepsilon(x) \tag{4}$$

where we have used the notation $\check{g}(y) = g(-y)$ and $g_\varepsilon(y) = g(y/\varepsilon)/\varepsilon^n$. By Young's inequality $W_g f(.,\varepsilon)$ is in L^p for all $\varepsilon > 0$ and W_g defines a continuous operator on this space for each ε.

If $g \in C_c(\mathbb{R}^m)$ we can define $W_g f$ for $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^m)$ directly by the same formula (4). If $g \in \mathcal{S}_0(\mathbb{R}^m)$ then W_g can be extended to $\mathcal{S}'(\mathbb{R}^m)$ as the adjoint of the wavelet synthesis (cf. [2], Ch. 1, Sects. 24, 25, and 30) or directly by $\mathcal{S}'$-$\mathcal{S}$-convolution in formula (4).

Remark 3. If f is a polynomial and $g \in \mathcal{S}_0$ it is easy to see that $W_g f = 0$. In fact, f, g, and $W_g f$ are in $\mathcal{S}'$ and $(\widehat{W_g f(.,\varepsilon)}) = \widehat{f}\widehat{\bar{g}}(\varepsilon.)$. Since g is in $\mathcal{S}_0$ the Fourier transform $\widehat{g}(\varepsilon.)$ is smooth and vanishes of infinite order at 0. But $\widehat{f}$ has to be a linear combination of derivatives of δ_0 implying $\widehat{f}\widehat{\bar{g}}(\varepsilon.) = 0$. Therefore the wavelet transform 'is blind to polynomial parts' of the analyzed function (or distribution) f. In terms of geophysical modelling this means that a polynomially varying background medium is filtered out automatically.

2.3 Wavelets from mollifiers

The Zygmund class characterization in Theorem 1 (and remark 2,(v)) used asymptotic estimates of scaled smoothings of the distribution which resembles typical mollifier constructions in Colombeau theory. In this subsection we relate this in turn directly to the wavelet transform obtaining the well-known wavelet characterization of Zygmund spaces.

Let $\chi \in \mathcal{S}(\mathbb{R}^m)$ with $\int \chi = 1$ and define the function μ by

$$\overline{\check{\mu}(x)} := m\chi(x) + \langle x | \operatorname{grad} \chi(x) \rangle \, . \tag{5}$$

Then μ is in $\mathcal{S}(\mathbb{R}^m)$ and is a wavelet since a simple integration by parts shows that

$$\begin{aligned} (-1)^{|\alpha|} \overline{\int \mu(x) x^\alpha \, dx} &= \int \bar{\check{\mu}}(x)(-x)^\alpha \, dx \\ &= (-1)^{|\alpha|+1} \int x^\alpha \chi(x)\, dx \sum_{j=1}^m \alpha_j = (-1)^{|\alpha|+1} |\alpha| \int x^\alpha \chi(x)\, dx \; . \end{aligned}$$

$\int \mu = 0$ and if $|\alpha| > 0$ we have $\int x^\alpha \mu(x)\, dx = 0$ if and only if $\int x^\alpha \chi(x)\, dx = 0$. Therefore μ defined by (5) is a wavelet of order N if and only if the mollifier χ has vanishing moments of order $1 \le |\alpha| \le N$.

Furthermore, by straightforward computation, we have

$$(\bar{\check{\mu}})_\varepsilon(x) = -\varepsilon \frac{d}{d\varepsilon}\big(\chi_\varepsilon(x)\big) \tag{6}$$

yielding an alternative of (5) in the form $\bar{\check{\mu}}(x) = -\frac{d}{d\varepsilon}\big(\chi_\varepsilon(x)\big)\,|_{\varepsilon=1}$.

If $(\varphi,\psi) \in \mathcal{M}(\mathbb{R}^m)$ arbitrary and χ, μ are the unique Schwartz functions such that $\widehat{\chi} = \varphi$ and $\widehat{\mu} = \psi$, then straightforward computation shows that μ and χ satisfy the relation (5). Therefore since μ is then a real valued and even wavelet we have for $u \in \mathcal{S}'$

$$\psi(tD)u(x) = t^m \bar{\check{\mu}}(\frac{\cdot}{t}) * u(x) = W_\mu u(x,t)\ .$$

Hence the distributions u in the Zygmund class $C^s_*(\mathbb{R}^m)$ can be characterized in terms of a wavelet transform and a smoothing pseudodifferential operator by $\|\varphi(D)u\|_{\mathrm{L}^\infty} < \infty$ and $\sup_{0<t<1}\Big(t^{-s}\|W_\mu u(.,t)\|_{\mathrm{L}^\infty}\Big) < \infty$. We have shown

Theorem 4. Let $(\widehat{\chi},\widehat{\mu}) \in \mathcal{M}(\mathbb{R}^m)$. A distribution $u \in \mathcal{S}'(\mathbb{R}^m)$ belongs to the Zygmund class $C^s_*(\mathbb{R}^m)$ if and only if

$$\|u * \chi\|_{\mathrm{L}^\infty} < \infty \quad \text{and} \quad \|W_\mu u(.,r)\|_{\mathrm{L}^\infty} = O(r^s) \ \ (r \to 0). \tag{7}$$

Remark 5. (i) Observe that the condition on $\widehat{\chi}$ implies that χ and hence μ can never have compact support. If this characterization is to be used in a theory of Zygmund regularity detection within Colombeau algebras one has to allow for mollifiers of this kind in the corresponding embedding procedures. This is the issue of the following subsection. Nevertheless we note here that according to remarks in [5, (2.2) and (3.1)] and, more precisely, in [9, Ch.3] the restrictions on the wavelet itself in a characterization of type (7) may be considerably relaxed — depending on the generality one wishes to allow for the analyzed distribution u. However, in case $m = 1$ and u a function a flexible and direct characterization (due to Holschneider and Tchamitchian) can be found in [1], Sect. 2.9, or [2], Sect. 4.2.

(ii) There are more refined results in the spirit of the above theorem describing local Hölder (Zygmund) regularity by growth properties of the wavelet transform (cf. in particular [2], Sect. 4.2, [6], and [5]).

(iii) The counterpart of (7) for $\mathrm{L}^1_{\mathrm{loc}}$-functions in terms of (discrete) multiresolution approximations is [8], Sect. 6.4, Thm. 5.

3 Colombeau modelling and wavelet transform

3.1 Embedding of temperate distributions

We consider a variant of the Colombeau embedding $\iota_\chi^\gamma : \mathcal{D}'(\mathbb{R}^m) \to \mathcal{G}(\mathbb{R}^m)$ that was discussed in [4], subsect. 3.2. As indicated in remark 5,(i) we need to allow for mollifiers with noncompact support in order to gain the flexibility of using wavelet-type arguments for the extraction of regularity properties from asymptotic estimates. On the side of the embedded distributions this forces us to restrict to $\mathcal{S}'$, a space still large enough for the geophysically motivated coefficients in model PDEs.

Recall ([4], Def. 11) that an admissible scaling is defined to be a continuous function $\gamma : (0,1) \to \mathbb{R}_+$ such that $\gamma(r) = O(1/r)$, $\gamma(r) \to \infty$, and $\gamma(sr) = O(\gamma(r))$ if $0 < s < 1$ (fixed) as $r \to 0$.

Definition 6. Let γ be an admissible scaling, $\chi \in \mathcal{S}(\mathbb{R}^m)$ with $\int \chi = 1$, then we define $\iota_\chi^\gamma : \mathcal{S}'(\mathbb{R}^m) \to \mathcal{G}(\mathbb{R}^m)$ by

$$\iota_\chi^\gamma(u) = \mathrm{cl}[(u * \chi^\gamma(\phi, .))_{\phi \in \mathcal{A}_0(\mathbb{R}^m)}] \qquad u \in \mathcal{S}'(\mathbb{R}^m) \tag{8}$$

where

$$\chi^\gamma(\phi, x) = \gamma(l(\phi_0))^m \chi(\gamma(l(\phi_0))x) \quad \text{if } \phi = \phi_0 \otimes \cdots \otimes \phi_0 \quad \text{with } \phi_0 \in \mathcal{A}_0. \tag{9}$$

ι_χ^γ is well-defined since $(\phi, x) \to u * \chi^\gamma(\phi, x)$ is clearly moderate and negligibility is preserved under this scaled convolution. By abuse of notation we will write $\iota_\chi^\gamma(u)(\phi, x)$ for the standard representative of $\iota_\chi(u)$.

The following statements describe properties of such a modelling procedure resembling the original properties used by M. Oberguggenberger in [10], Prop.1.5, to ensure unique solvability of symmetric hyperbolic systems of PDEs (cf. [10, 7]). The definition of Colombeau functions of logarithmic and bounded type is given in [11], Def. 19.2, the variation used below is an obvious extension.

Proposition 7.

(i) $\iota_\chi^\gamma : \mathcal{S}'(\mathbb{R}^m) \to \mathcal{G}(\mathbb{R}^m)$ is linear, injective, and commutes with partial derivatives.

(ii) $\forall u \in \mathcal{S}'(\mathbb{R}^m)$: $\iota_\chi^\gamma(u) \approx u$.

(iii) If $u \in W_\infty^{-1}(\mathbb{R}^m)$ then $\iota_\chi^\gamma(u)$ is of *γ-type*, i.e., there is $N \in \mathbb{N}_0$ such that for all $\phi \in \mathcal{A}_N(\mathbb{R}^m)$ there exist $C > 0$ and $1 > \eta > 0$:

$$\sup_{y \in \mathbb{R}^m} |\iota_\chi^\gamma(u)(\phi_\varepsilon, y)| \le N\gamma(C\varepsilon) \quad 0 < \varepsilon < \eta. \tag{10}$$

(iv) If $u \in \mathrm{L}^\infty(\mathbb{R}^m)$ then $\iota_\chi^\gamma(u)$ is of bounded type and its first order derivatives are of γ-type.

Proof. ad (i),(ii): Is clear from $\chi_\varepsilon := \chi^\gamma(\phi_\varepsilon, .) \to \delta$ in $\mathcal{S}'$ as $\varepsilon \to 0$ and the convolution formula.

ad (iii): Although this involves only marginal changes in the proof of [10], Prop. 1.5(i), we recall it here to make the presentation more self-contained.

Let $u = u_0 + \sum_{j=1}^m \partial_j u_j$ with $u_j \in \mathrm{L}^\infty$ $(j = 0, \ldots, m)$ then with $\gamma_\varepsilon := \gamma(\varepsilon l(\phi_0))$

$$\begin{aligned}
|u * \chi_\varepsilon(x)| &\le \|u_0 * \chi_\varepsilon\|_{\mathrm{L}^\infty} + \sum_{j=1}^m \|u_j * \partial_j(\chi_\varepsilon)\|_{\mathrm{L}^\infty} \\
&\le \|u_0\|_{\mathrm{L}^\infty}\|\chi\|_{\mathrm{L}^1} + \gamma_\varepsilon \sum_{j=1}^m \|u_j\|_{\mathrm{L}^\infty}\|\partial_j\chi\|_{\mathrm{L}^1} \\
&= \gamma_\varepsilon\big(\frac{\|u_0\|_{\mathrm{L}^\infty}\|\chi\|_{\mathrm{L}^1}}{\gamma_\varepsilon} + \sum_{j=1}^m \|u_j\|_{\mathrm{L}^\infty}\|\partial_j\chi\|_{\mathrm{L}^1}\big)
\end{aligned}$$

where the expression within brackets on the r.h.s. is bounded by some constant M, dependent on u and χ only but independent of ϕ, as soon as $\varepsilon < \eta$ with η chosen appropriately (and dependent on M, u, χ, and ϕ). Therefore the assertion is proved by putting $N \ge M$ and $C = l(\phi_0)$.

ad (iv): is proved by similar reasoning $\square$

In particular, we can model a fairly large class of distributions as Colombeau functions of logarithmic growth (or log-type) thereby ensuring unique solvability of hyperbolic PDEs incorporating such as coefficients.

Corollary 8.

(i) If $\gamma(\varepsilon) = \log(1/\varepsilon)$ then $\iota_\chi^\gamma(W^{-1,\infty}) \subseteq \{U \in \mathcal{G} \mid U \text{ is of log-type }\}$ and

$\iota_\chi^\gamma(\mathrm{L}^\infty) \subseteq \{U \in \mathcal{G} \mid U \text{ of bounded type and } \partial^\alpha U \text{ of log-type for } |\alpha| = 1\}$.

(ii) If $u \in W^{-k,\infty}(\mathbb{R}^m)$ for $k \in \mathbb{N}_0$ then $\iota_\chi^\gamma(u)$ is of γ^k-type. In particular, there is an admissible scaling γ such that $\iota_\chi^\gamma(u)$ and all first order derivatives $\partial_j \iota_\chi^\gamma(u)$ $(j = 1, \ldots, m)$ are of log-type.

3.2 Wave front sets under the embedding

One of the most important properties of the embedding procedure introduced in [4] was its faithfulness with respect to the microlocal properties if 'appropriately measured' in terms of the set of γ-regular Colombeau functions $\mathcal{G}_\gamma^\infty(\mathbb{R}^m)$ ([4], Def. 11). But there the proof of this microlocal invariance property heavily used the compact support property of the standard mollifier χ which is no longer true in the current situation. In this subsection we show how to extend the invariance result to the new embedding procedure defined above.

Theorem 9. Let $w \in \mathcal{S}'(\mathbb{R}^m)$, γ an admissible scaling, and $\chi \in \mathcal{S}(\mathbb{R}^m)$ with $\int \chi = 1$ then

$$WF_g^\gamma(\iota_\chi^\gamma(w)) = WF(w). \tag{11}$$

Proof. The necessary changes in the proof of [4], Thm. 15, are minimal once we established the following

Lemma 10. If $\varphi \in \mathcal{D}(\mathbb{R}^m)$ and $v \in \mathcal{S}'(\mathbb{R}^m)$ with $\mathrm{supp}(\varphi) \cap \mathrm{supp}(v) = \emptyset$ then $\varphi \cdot \iota_\chi^\gamma(v) \in \mathcal{G}_\gamma^\infty$.

Proof. Using the short-hand notation $\chi_\varepsilon = \chi^\gamma(\phi_\varepsilon, .)$ and $\gamma_\varepsilon = \gamma(\varepsilon l(\phi_0))$ we have

$$\partial^\beta\big(\varphi(v * \chi_\varepsilon)\big)(x) = \gamma_\varepsilon^m \sum_{\alpha \le \beta} \binom{\beta}{\alpha} \partial^{\beta-\alpha}\varphi(x)\, \gamma_\varepsilon^{|\alpha|} \left\langle v, \partial^\alpha \chi(\gamma_\varepsilon(x - .))\right\rangle .$$

Hence we need to estimate terms of the form $\gamma_\varepsilon^{|\alpha|} \left\langle v, \partial^\alpha \chi(\gamma_\varepsilon(x - .))\right\rangle$ when $x \in \mathrm{supp}(\varphi) =: K$. Let S be a closed set satisfying $\mathrm{supp}(v) \subset S \subset \mathbb{R}^m \setminus K$ and put $d = \mathrm{dist}(S, K) > 0$. Since v is a temperate distribution there is $N \in \mathbb{N}$ and $C > 0$ such that

$$\gamma_\varepsilon^{|\alpha|} |\langle v, \partial^\alpha \chi(\gamma_\varepsilon(x - .))\rangle| \le C \gamma_\varepsilon^{|\alpha|} \sum_{|\sigma| \le N} \sup_{y \in S} \left|\partial^\sigma\big(\partial^\alpha \chi(\gamma_\varepsilon(x - y))\big)\right| .$$

$\chi \in \mathcal{S}$ implies that each term in the sum on the right-hand side can be estimated for arbitrary $k \in \mathbb{N}$ by

$$\sup_{y \in S} \left|\partial^{\sigma+\alpha} \chi(\gamma_\varepsilon(x - y))\right) | \gamma_\varepsilon^{|\sigma|} \le \gamma_\varepsilon^{|\sigma|} \sup_{y \in S} C_k (1 + \gamma_\varepsilon |x - y|)^{-k} \le C_k' \gamma_\varepsilon^{|\sigma| - k} / d^k$$

if x varies in K. Since $|\alpha| + |\sigma| \le |\beta| + N$ we obtain

$$\|\partial^\beta\big(\varphi(v * \chi_\varepsilon)\big)\|_{\mathrm{L}^\infty} \le C' \gamma_\varepsilon^{m+N+|\beta|-k}$$

with a constant C' depending on k, v, φ, d, and χ but k still arbitrary. Choosing $k = |\beta|$, for example, we conclude that $\varphi \cdot \iota_\chi^\gamma(v)$ has a uniform γ_ε-growth over all orders of derivatives. Hence it is a γ_ε-regular Colombeau function. ☐

Referring to the proof (and the notation) of [4], Thm. 15, we may now finish the proof of the theorem simply by carrying out the following slight changes in the two steps of that proof.

Ad step 1: Choose $\psi \in \mathcal{D}$ such that $\psi = 1$ in a neighborhood of $\mathrm{supp}(\varphi)$ and write

$$\varphi(w * \chi_\varepsilon) = \varphi((\psi w) * \chi_\varepsilon) + \varphi(((1 - \psi) w) * \chi_\varepsilon\ .$$

The first term on the right can be estimated by the same methods as in [4] and the second term is γ-regular by the lemma above.

Ad step 2: Rewrite

$$\varphi w = \varphi\psi w = \varphi(\psi w - (\psi w) * \chi_\varepsilon) + \varphi((\psi w) * \chi_\varepsilon)$$

and observe that the reasoning of [4] is applicable since $\Sigma_g^\gamma(\varphi\, \iota_\chi^\gamma(\psi w)) \subseteq \Sigma_g^\gamma(\varphi\, \iota_\chi^\gamma(w))$ by the above lemma.

□

3.3 The modelling procedure and wavelet transforms

Simple wavelet-mollifier correspondences as in subsection 2.3 allow us to rewrite the Colombeau modelling procedure and hence prepare for the detection of original Zygmund regularity in terms of growth properties in the scaling parameters.

A first version describes directly ι_χ^γ but involves an additional nonhomogeneous term.

Lemma 11. If $\chi \in \mathcal{S}(\mathbb{R}^m)$ has the properties $\int \chi = 1$ and $\int x^\alpha \chi(x) dx = 0$ $(0 < |\alpha| \leq N)$ then $\bar{\check{\mu}} = -\frac{d}{d\varepsilon}(\chi_\varepsilon) \mid_{\varepsilon=1}$ defines a wavelet of order N and we have for any $f \in \mathcal{S}'(\mathbb{R}^m)$

$$\iota_\chi^\gamma(f)(\phi, x) = f * \chi(x) + \int\limits_{1/\gamma(l(\phi))}^{1} W_\mu f(x, r)\, \frac{dr}{r}\ . \tag{12}$$

Proof. Let $\varepsilon > 0$ then eq. (6) implies $W_\mu f(x, \varepsilon) = f * (\bar{\check{\mu}})_\varepsilon(x) = -\varepsilon \frac{d}{d\varepsilon}\big(f * \chi_\varepsilon(x)\big)$ and integration with respect to ε from $1/\gamma(l(\phi))$ to 1 yields

$$-\int\limits_{1/\gamma(l(\phi))}^{1} W_\mu f(x, \varepsilon) \frac{d\varepsilon}{\varepsilon} = f * \chi(x) - \iota_\chi^\gamma(f)(\phi, x)\ .$$

□

A more direct mollifier wavelet correspondence is possible via derivatives of ι_χ^γ instead.

Lemma 12. If $\chi \in \mathcal{S}(\mathbb{R}^m)$ with $\int \chi = 1$ then for any $\alpha \in \mathbb{N}_0^n$ with $|\alpha| > 0$

$$\chi_\alpha(x) = \overline{(\partial^\alpha \chi)\check{}\,(x)} \tag{13}$$

is a wavelet of order $|\alpha| - 1$ and for any $f \in \mathcal{S}'(\mathbb{R}^m)$ we have

$$\partial^\alpha \iota_\chi^\gamma(f)(\phi, x) = \gamma(l(\phi))^{|\alpha|}\, W_{\chi_\alpha} f(x, \frac{1}{\gamma(l(\phi))})\ . \tag{14}$$

Proof. Let $|\beta| < |\alpha|$ then $\int x^\beta D^\alpha \chi(x)\,dx = (-D)^\beta(\xi^\alpha \widehat{\chi}(\xi))\ |_{\xi=0} = 0$ which proves the first assertion. The second assertion follows from

$$\partial^\alpha \iota_\chi^\gamma(f)(\phi) = \gamma^{|\alpha|+m} f * \partial^\alpha \chi(\gamma .) = \gamma^{|\alpha|} f * \overline{\left(\gamma^m (\partial^\alpha \bar{\chi})\check{}\,(\gamma .)\right)}\check{}$$

with the short-hand notation $\gamma = \gamma(l(\phi))$. □

Both lemmas 11 and 12 may be used to translate (global) Zygmund regularity of the modeled (embedded) distribution f via Thm. 7 into asymptotic growth properties with respect to the regularization parameter. To what extent this can be utilized to develop a faithful and completely intrinsic Zygmund regularity theory of Colombeau functions may be subject of future research.

4 Zygmund regularity of Colombeau functions: the one-dimensional case

If we combine the basic ideas of the Zygmund class characterization in 2.3 with the simple observations in 3.3 we are naturally lead to define a corresponding regularity notion intrinsically in Colombeau algebras as follows.

Definition 13. Let γ be an admissible scaling function and s be a real number. A Colombeau function $U \in \mathcal{G}(\mathbb{R}^m)$ is said to be *globally of γ-Zygmund regularity s* if for all $\alpha \in \mathbb{N}_0^m$ there is $M \in \mathbb{N}_0$ such that for all $\phi \in \mathcal{A}_M(\mathbb{R}^m)$ we can find positive constants C and η such that

$$(15) \qquad |\partial^\alpha U(\phi_\varepsilon, x)| \le \begin{cases} C & \text{if } |\alpha| < s \\ C\ \gamma_\varepsilon^{|\alpha|-s} & \text{if } |\alpha| \ge s \end{cases} \qquad x \in \mathbb{R}^m, 0 < \varepsilon < \eta.$$

The set of all (globally) γ-Zygmund regular Colombeau functions of order s will be denoted by $\mathcal{G}^s_{*,\gamma}(\mathbb{R}^m)$.

A detailed analysis of $\mathcal{G}^s_{*,\gamma}$ in arbitrary space dimensions and not necessarily positive regularity s will appear elsewhere. Here, as an illustration, we briefly study the case $m = 1$ and $s > 0$ in some detail. Concerning applications to PDEs this would mean that we are allowing for media of typical fractal nature varying continuously in one space dimension. For example one may think of a coefficient function f in $C^s_*(\mathbb{R})$ to appear in the following ways.

Example 14. (i) Let f be constant outside some interval $(-K, K)$ and equal to a typical trajectory of Brownian motion in $[-K, K]$; it is well-known that with probability 1 those trajectories are in $C^s_*(\mathbb{R})$ whenever $s < 1/2$. This is proved, e.g., in [2], Sect. 4.4, elegantly by wavelet transform methods.

(ii) We refer to [16], Sect. V.3, for notions and notation in this example. Then similarly to the above one can set $f = 0$ in $(-\infty, 0]$, $f = 1$ in $[2\pi, \infty)$ and in $[0, 2\pi]$ let f be Lebesgue's singular function associated with a Cantor-type set of order $d \in \mathbb{N}$ with (constant) dissection ratio $0 < \xi < 1/2$. Then f belongs to $C^s_*(\mathbb{R})$ with $s = \log(d+1)/|\log(\xi)|$. (The classical triadic Cantor set corresponds to the case $d = 2$ and $\xi = 1/3$.)

We have already seen that the Colombeau embedding does not change the microlocal structure (i.e., the γ-wave front set) of the original distribution. We will show now that also the refined Zygmund regularity information is accurately preserved. If $n \in \mathbb{N}_0$ we denote by $C^n_b(\mathbb{R})$ the set of all n times continuously differentiable functions with the derivatives up to order n bounded. Note that $C^n_b(\mathbb{R})$ is a strict superset of $C^{n+1}_*(\mathbb{R})$.

Theorem 15. Let $\chi \in \mathcal{A}_0(\mathbb{R})$ and $s > 0$. Define $n \in \mathbb{N}_0$ such that $n < s \leq n+1$ then we have

$$\iota^\gamma_\chi(C^n_b(\mathbb{R})) \cap \mathcal{G}^s_{*,\gamma}(\mathbb{R}) = \iota^\gamma_\chi(C^s_*(\mathbb{R})). \tag{16}$$

In other words, in case $0 < s < 1$ we can precisely identify those Colombeau functions that arise from the Zygmund class of order s within all embedded bounded continuous functions.

Proof. We use the characterizations in [1], Thms. 2.9.1 and 2.9.2 and the remarks on p. 48 following those; choosing a smooth compactly supported wavelet g of order n we may therefore state the following[1]: $f \in C^n_b(\mathbb{R})$ belongs to $C^s_*(\mathbb{R})$ if and only if there is $C > 0$ such that

$$|W_g f(x, r)| \leq C r^s \quad \text{for all } x. \tag{17}$$

Now the proof is straightforward. First let $f \in C^s_*(\mathbb{R})$. If $|\alpha| - 1 < n$ then $|\partial^\alpha \iota^\gamma_\chi(f)| = |\iota^\gamma_\chi(\partial^\alpha f)| \leq \|\partial^\alpha f\|_{\mathrm{L}^\infty} \|\chi\|_{\mathrm{L}^1}$ by Young's inequality. If $|\alpha| > n$ we use lemma 12 and set $\gamma_\varepsilon = \gamma(\varepsilon l(\phi))$ to obtain

$$\partial^\alpha \iota^\gamma_\chi(f)(\phi_\varepsilon, x) = \gamma_\varepsilon^{|\alpha|} W_{\chi_\alpha} f(x, \gamma_\varepsilon^{-1})$$

where χ_α is a wavelet of order at least n. Hence (17) gives an upper bound $C\gamma_\varepsilon^{|\alpha|-s}$ uniformly in x. Hence (15) follows.

Finally, if we know that $f \in C^n_b(\mathbb{R})$ and $\iota^\gamma_\chi(f) \in \mathcal{G}^s_{*,\gamma}(\mathbb{R})$ then combination of (15) and lemma 12 gives if $|\alpha| \geq s$

$$|\gamma_\varepsilon^{|\alpha|} W_{\chi_\alpha} f(x, \gamma_\varepsilon^{-1})| = |\partial^\alpha \iota^\gamma_\chi(f)(\phi_\varepsilon, x)| \leq C\gamma_\varepsilon^{|\alpha|-s}$$

uniformly in x. Hence another application of (17) proves the assertion. $\square$

[1] Note that we do not use the wavelet scaling convention adapted to L^2-spaces here

References

[1] I. Daubechies. *Ten lectures on wavelets.* Number 61 in CBMS. SIAM, Philadelphia, 1992.

[2] M. Holschneider. *Wavelets: an analysis tool.* Oxford University Press, New York, 1995.

[3] L. Hörmander. *Lectures on Nonlinear Hyperbolic Differential Equations.* Springer-Verlag, Berlin Heidelberg, 1997.

[4] G. Hörmann and M. V. de Hoop. Microlocal analysis and global solutions of some hyperbolic equations with discontinuous coefficients. *to appear in Acta Appl. Math.*, 2001.

[5] S. Jaffard. Multifractal formalism for functions, part I: results valid for all functions. *SIAM J. Math. Anal.*, 28(4):944–970, 1997.

[6] S. Jaffard and Y. Meyer. Wavelet methods for pointwise regularity and local oscillations of functions. *Mem. Amer. Math. Soc.*, 123(587), 1996.

[7] F. Lafon and M. Oberguggenberger. Generalized solutions to symmetric hyperbolic systems with discontinuous coefficients: the multidimensional case. *J. Math. Anal. Appl.*, 160:93–106, 1991.

[8] Y. Meyer. *Wavelets and Operators.* Cambridge studies in advanced mathematics 37. Cambridge University Press, Cambridge, 1992.

[9] Y. Meyer. *Wavelets, Vibrations and Scalings.* CRM Monograph Series 9. American Mathematical Society, Providence, 1998.

[10] M. Oberguggenberger. Hyperbolic systems with discontinuous coefficients: generalized solutions and a transmission problem in acoustics. *J. Math. Anal. Appl.*, 142:452–467, 1989.

[11] M. Oberguggenberger. *Multiplication of Distributions and Applications to Partial Differential Equations.* Pitman Research Notes in Mathematics 259. Longman Scientific & Technical, 1992.

[12] H. Triebel. Characterizations of Besov-Hardy-Sobolev spaces via harmonic functions, temperatures, and related means. *Jour. Approx. Theory*, 35:275–297, 1982.

[13] H. Triebel. *Theory of Function Spaces.* Akademische Verlagsgesellschaft Geest & Portig and Birkhäuser Verlag, Leipzig and Basel, 1983.

[14] H. Triebel. Characterizations of Besov-Hardy-Sobolev spaces: a unified approach. *Jour. Approx. Theory*, 52:162–203, 1988.

[15] H. Triebel. *Theory of Function Spaces II.* Birkhäuser Verlag, Basel, 1992.

[16] A. Zygmund. *Trigonometric series, volumes I and II*. Cambridge University Press, London, New York, 1968.